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On Analytic Review of the Binomial Theorem and its Generalized Results with Some Extension to Series

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Abstract

The Binomial theorem in this work was sufficiently reviewed before it was used in the development of the trinomial theorem. Subsequent upon this was the continued extension of this result to the nth finite order, the continuous expansion of which resulted in the complicated multinomial theorem, the ideas of which were used in the last section to analytically develop different multinomial series and their convergence results.

Keywords:
Binomial expansion and Binomial theorem, Negative Multinomials, Multibinomial series.

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Introduction

Any expression such as (a+x) involving two terms is known as a binomial expression and thus $(a+x)^n$ is a binomial function and the statement of its expansion in powers of x is known as the binomial theorem (*n* is the index).
The method used in proving the binomial theorem for a positive integral index is known as the method of induction and in this method (which can be applied to various other theorems and problems), it is assumed that the expansion holds true for power n. Next it is proved by algebra that the result is true when n is replaced by (*n*+1). The expansion is then proved to be valid for *n*=2. This being the case, it will also be true for *n*=3, and therefore *n*=4, and so on for all positive integral values of *n*.

It is assumed that if *n* be a positive integer, then

$$(a+x)^n = a^n + {}^nC_1 a^{n-1}x + {}^nC_2 a^{n-2}x^2 + \cdots + {}^nC_{r-1}a^{n-r+1}x^{r-1} + {}^nC_r a^{n-r}x^r + \cdots + x^n \qquad (1.1)$$

Multiplying (1.1) by (a+x) and showing only necessary terms,

$$(a+x)^{n+1} = a^{n+1} + {}^nC_1 a^n x + {}^nC_2 a^{n-1}x^2 + \cdots + {}^nC_r a^{n-r+1}x^r + \cdots + ax^n + a^n x + {}^nC_1 a^{n-1}x^2 + \cdots + {}^nC_{r-1}a^{n-r}x^r + \cdots + x^{n+1}$$

$$= a^{n+1} + ({}^nC_1 + 1)a^n x + ({}^nC_2 + {}^nC_1)a^{n-1}x^2 + \cdots + ({}^nC_r + {}^nC_{r-1})a^{n-r+1}x^r + \cdots + x^{n+} \qquad (1.2)$$

Now, ${}^nC_1 + 1 = n + 1 = {}^{n+1}C_1$,

and ${}^nC_r + {}^nC_{r-1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$

$$= n!\left\{\frac{1}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}\right\}$$

$$= \frac{n!}{r!(n-r+1)!}\left\{\frac{(n-r+1)}{(n-r)!} + \frac{r!}{(r-1)!}\right\}$$

1 $$= \frac{n!}{r!(n-r+1)!}[(n-r+1)+r]$$

$$= \frac{(n+1).n!}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n-r+1)!} = {}^{n+1}C_r$$

Using these results (1.2) becomes [Hsu (1990)]

$$(a+x)^{n+1} = a^{n+1} + {}^{n+1}C_1 a^n x + {}^{n+1}C_2 a^{n-1} x^2 + \cdots + {}^{n+1}C_r a^{n-r+1} x^r + \cdots + x^{n+1} \qquad (1.3)$$

If n be replaced by $(n+1)$ in (1.1)
the equation (1.3) is obtained.
Thus, if the binomial theorem be true for the index n it must also be true for the index $(n+1)$. By algebra [Trench (2010)]

$$(a+x)^2 = a^2 + 2ax + x^2$$

$$= a^2 + {}^2C_1 ax + x^2 \qquad (2 = {}^2C_1)$$

$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$$

$$= a^3 + {}^3C_1 a^2 x + {}^3C_2 a x^2 + x^3 \qquad (3 = {}^3C_1 = {}^3C_2)$$

But these are the results of replacing n by 2 and 3 respectively in equation (1.1) and therefore the binomial theorem is true for n=2 and n=3.
[Tranta and Landa (1980)]
Hence, by statement above, since the theorem is true for n=3, it must be true for n=4 and since its true for n=4, it must be true for n=5, and so on for all positive integral values of n. Thus, for all positive integral values of n

$$(a+x)^n = a^n + {}^nC_1 a^{n-1} x + {}^nC_2 a^{n-2} x^2 + \cdots + {}^nC_r a^{n-r} x^r + \cdots + x^n$$

The notation ${}^{n}C_{r}$ is best reserved for the proof and the expansion of the binomial theorem should be remembered in the following equivalent form, since it also applies to the case when n is not a positive integer.

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \cdots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}a^{n-r}x^r$$

$$+\cdots$$

Note: When one part of the binomial expression is negative, it is advisable to consider the negative quantity as +x in $(a+x)^n$. In this manner the formula as stated can be used with all positive signs between the terms.

The binomial theorem can be modified to apply to a trinomial expression $(a+b+c)^n$ and further extended to apply to multinomial expression (more than three terms). In the case of the trinomial expression $(a+b+c)^n$ the quantity (b+c) is chosen as a single term initially and the expression to be expanded will be written as [Royden (2008)];

The quantities $(b+c)^2$,$(b+c)^3$ etc. will then each be expanded by the binomial theorem, and the required expansion obtained after simplification.

Note: If a, b or c be equal to unity, that term is generally kept as the single term and the remaining two combined as a single term.

We seek the formula for

(1). $\left[\sum_{i=1}^{m} a_i\right]^n$ called the positive multinomial i.e. for $n \geq 0$

(2.1) $\left[\sum_{i=1}^{m} a_i\right]^{-n}$ called the negative multinomial i.e. for $n \leq 0$

(2.1) $\left[\sum_{i=1}^{m} a_i\right]^{r=\frac{k}{n}}$ called the rational multinomial i.e. $r = \frac{k}{n}$, $k > 0$, $n \neq 0$

(2.1) $\left[\sum_{i=1}^{m} a_i\right]^{r=\frac{k}{n}}$ called the rational multinomial i.e.

$$r=\frac{k}{n},\ k>0,\ \ n\neq 0$$

To achieve this given any Binomial theorem, we can apply its method of expansion in the generation of a trinomial theorem. Continuing this extension finitely to the n^{th} order, we then develop the complicated multinomial theorem in its expanded form as seen in this work. In the same vein, the multibinomial theorem was also generated, the ideas of which were used in the last section to analytically develop different multinomial series and their convergence results.

We seek expansion formula for

$$\left[\sum_{i=1}^{m} a_i\right]^n \tag{1.1}$$

To start with, let $m=2$ then the above becomes

$$\left[\sum_{i=1}^{2} a_i\right]^n = \left[(a_1+a_2)\right]^n$$
$$= a_1^n + na_1^{n-1}a_2 + \frac{n(n-1)}{2}a^{n-2}a^2 + \ldots + \frac{n(n-1)(n-2)\quad(n-r+1)}{r!}a^{n-r}x^r + \ldots$$
$$= \sum_{r=0}^{n}\binom{n}{r}a_1^{n-r}a_2^r$$

Again let $m=3$, in (1.1) above, we have

$$\left[\sum_{i=1}^{3} a_i\right]^n = \left[(a_1+(a_2+a_3))\right]^n$$
$$= a_1^n + na_1^{n-1}(a_2+a_3) + \frac{n(n-1)}{2!}a_1^{n-2}*(a_2+a_3)^2 + \quad + \frac{n(n-1)(n-2)}{3!}a_1^{n-3}*(a_2+a_3)^3$$
$$+\frac{n(n-1)(n-2)(n-3)}{3!}a_1^{n-4}*(a_2+a_3)^4 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}a^{n-r}(a_2+a_3)^r + \ldots$$

$$+\frac{n(n-1)(n-2)(n-3)}{3!}a_1^{n-4}*(a_2+a_3)^4+$$

$$+\frac{n(n-1)(n-2)(n-r+1)}{r!}\quad a^n\ (a_2^r)+a_3^r\ +$$

$$\left[\left(a_1+\left(a_2+a_3\right)\right)\right]^n = a_1^n + na_1^{n-1}a_2 + na_1^{n-1}a_3 + \frac{n(n-1)}{2!}a_1^{n-2}a_2^2 + \frac{n(n-1)}{2!}a_1^{n-2}2a_2a_3^2$$

$$\frac{n(n-1)}{2!}a_1^{n-2}a_2a_3^2 + \frac{n(n-1)(n-2)}{3!}a_1^{n-3}a_2^3 + \frac{n(n-1)(n-2)}{3!}a_1^{n-3}*3a_2^2a_3$$

$$+\frac{n(n-1)(n-2)}{3!}a_1^{n-3}*3a_2^2a_3 + \frac{n(n-1)(n-2)}{3!}a_1^{n-3}3a_2a_3^2 + \frac{n(n-1)(n-2)}{3!}a_1^{n-3}a_3^2$$

$$+\frac{n(n-1)(n-2)(n-3)}{4!}a_1^{n-4}a_2^4 + \frac{n(n-1)(n-2)(n-3)}{4!}a_1^{n-4}*4a_2^3a_3$$

$$+\frac{n(n-1)(n-2)(n-3)}{4!}a_1^{n-4}*6a_2a_3^2 + \frac{n(n-1)(n-2)(n-3)}{4!}a_1^{n-4}*4a_2a_3$$

$$+\frac{n(n-1)(n-2)(n-3)}{4!}a_1^{n-4}a_3^4 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}a_1^{n-r}(a_2+a_3)^r + \ldots$$

$$= \sum_{r=0}^{n}\binom{n}{r}a_1^{n-r}a_2^r(a_2+a_3)^r == \sum_{r=0}^{n}\sum_{r_d}^{n_d}\binom{n}{r}a_1^{n-r}a_2^r\binom{n_d}{r_d}a_2^{n_d}+a_3^{r_d};\ n_d=r,\ r_d=k$$

Also let $4m$=in (1.1), we have

$$\left[(a_1+a_2)+(a_3+a_4)\right]^n$$

$$=(a_1+a_2)^n + n(a_1+a_2)^{n-1}(a_3+a_4) + \frac{n(n-1)}{2!}(a_1+a_2)^{n-2}(a_3+a_4)^2$$

$$+\frac{n(n-1)(n-2)}{3!}(a_1+a_2)^{n-3}(a_3+a_4)^3 + \frac{n(n-1)(n-2)(n-3)}{4!}(a_1+a_2)^{n-4}(a_3+a_4)^4 + \cdots$$

$$=\sum_{r=0}^{n}\binom{n}{r}(a_1+a_2)^{n-r}(a_3+a_4)^r$$

$$=\left(a_1^n + na_1^{n-1}a_2 + \frac{n(n-1)}{2!}a_1^{n-2}a_2^2 + \frac{n(n-1)(n-2)}{3!}a_1^{n-3}a_2^3 + \frac{n(n-1)(n-2)(n-3)}{4!}a_1^{n-4}a_2^4 + \cdots\right)$$

$$+\left[\left(na_1^{n-1}a_3 + n(n-1)a_1^{n-2}a_2a_3 + \frac{n(n-1)(n-2)}{2!}a_1^{n-3}a_2^2a_3 + \frac{n(n-1)(n-2)(n-3)}{4!}a_1^{n-5}a_2^4a_3 + \cdots\right)\right.$$

$$+\left(na_1^{n-1}a_4 + n(n-1)a_1^{n-2}a_2a_4 + \frac{n(n-1)(n-2)}{2!}a_1^{n-3}a_2^2a_4 + \frac{n(n-1)(n-2)(n-3)}{3!}a_1^{n-3}a_2^2a_4\right.$$

$$+\frac{n(n-1)(n-2)(n-3)}{3!}a_1 a_2^{3} a_4+\frac{n(n-1)(n-2)(n-3)}{4!}a_1^{n-5}a_2^{4}a_4+\cdots\Big)\Big]$$

$$+\Bigg[\Bigg(\frac{n(n-1)}{2!}a_1^{n-2}a_3^{2}+\frac{n(n-1)(n-2)}{2!}a_1^{n-3}a_2a_3^{2}+\frac{n(n-1)}{2!}\frac{(n-2)(n-3)}{2!}a_1^{n-4}a_2^{2}a_3^{2}$$

$$+\frac{n(n-1)}{2!}\frac{(n-2)(n-3)(n-4)}{3!}a_1^{n-5}a_2^{3}a_3^{2}+\frac{n(n-1)}{2!}\frac{(n-2)(n-3)(n-4)(n-5)}{4!}a_1^{n-6}a_2^{4}a_3^{2}+\cdots\Bigg)$$

$$+\frac{n(n-1)}{2!}a_1^{n-2}*2a_3a_4+\frac{n(n-1)(n-2)}{2!}a_1^{n-3}a_2a_3a_4*2$$

$$+\frac{n(n-1)}{2!}\frac{(n-2)(n-3)(n-4)}{3!}a_1^{n-5}a_2^{3}a_3a_4*2$$

$$+\frac{n(n-1)}{2!}\frac{(n-2)(n-3)(n-4)(n-5)}{4!}a_1^{n-5}*a_2^{4}a_3a_4*2+\cdots\Bigg)$$

$$+\Bigg(\frac{n(n-1)}{2!}a_1^{n-2}a_4^{2}+\frac{n(n-1)}{2!}(n-2)a_1^{n-3}a_2a_4^{2}+\frac{n(n-1)}{2!}\frac{(n-2)(n-3)}{2!}a_1^{n-4}a_2^{2}a_4^{2}$$

$$+\frac{n(n-1)}{2!}\frac{(n-2)(n-3)(n-4)}{3!}a_1^{n-5}a_2^{3}a_4^{2}+\frac{n(n-1)}{2!}\frac{(n-2)(n-3)(n-4)(n-5)}{4!}a_1^{n-6}a_2^{4}a_4^{2}+\cdots\Bigg)\Bigg]$$

$$+\Bigg[\Bigg(\frac{n(n-1)(n-2)}{3!}a_1^{n-3}a_3^{3}+\frac{n(n-1)(n-2)(n-3)}{3!}a_1^{n-4}a_2a_3^{3}$$

$$+\frac{n(n-1)(n-2)}{3!}\frac{(n-3)(n-4)}{2!}a_1^{n-5}a_2^{2}a_3^{3}$$

$$+\frac{n(n-1)(n-2)}{3!}\frac{(n-3)(n-4)(n-5)}{3!}a_1^{n-6}a_2^{3}a_3^{3}$$

$$+\frac{n(n-1)(n-2)}{3!}\frac{(n-4)(n-5)}{4!}a_1^{n-7}a_2^{3}a_3^{3}+\cdots\Bigg)$$

$$+\Bigg(\frac{n(n-1)(n-2)}{3!}a_1^{n-3}*3a_3^{2}a_4+\frac{n(n-1)(n-2)}{3!}(n-3)a_1^{n-4}a_2*3a_2^{2}a_4$$

$$+\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{3!}a_1^{n-6}a_2^{2}*3a_3^{2}a_4$$

$$\frac{n(n-1)(n-2)}{3!}\frac{(n-3)(n-4)(n-5)(n-6)}{4!}3a_3^{2}a_4+\cdots\Bigg)$$

$$+\Bigg(\frac{n(n-1)(n-2)}{3!}a_1^{n-3}*3a_3a_4^{2}+\frac{n(n-1)(n-2)}{3!}(n-3)a_1^{n-4}a_2*3a_3a_4^{2}$$

$$+\frac{n(n-1)(n-2)}{3!}\frac{(n-3)(n-4)}{2!}a_1^{n-5}a_2^{2}*3a_3a_4^{2}+\frac{n(n-1)(n-2)}{3!}\frac{(n-3)(n-4)(n-5)}{3!}a_1^{n-6}a_2^{3}*3a_3a_4^{2}$$

$$+\frac{n(n-1)(n-2)}{3!}\frac{(n-3)(n-4)(n-5)(n-6)}{4!}a_1^{n-7}a_2^{4}*3a_3a_4^{2}\Bigg)$$

$$+\left(\frac{n(n-1)(n-2)}{3!}a_1a_4^3+\frac{n(n-1)(n-2)(n-3)}{3!}a_1^{n-4}a_2a_4^3\right.$$

$$+\frac{n(n-1)(n-2)}{3!}\frac{(n-3)(n-4)}{2!}a_1^{n-5}a_2^2a_4^3+\frac{n(n-1)(n-2)}{3!}(n-3)(n-4)(n-5)a_1^{n-6}a_2^3a_4^3$$

$$\left.\frac{n(n-1)(n-2)}{3!}\frac{(n-3)(n-4)(n-5)(n-6)}{4!}a_1^{n-7}a_2^3a_4^3+\cdots\right)$$

$$+\left[\left(\frac{n(n-1)(n-2)(n-3)}{3!}a_1^{n-4}a_3^4+\frac{n(n-1)(n-2)(n-3)(n-4)}{4!}a_1^{n-5}a_2a_3^4\right.\right.$$

$$+\frac{n(n-1)(n-2)(n-3)}{4!}\frac{(n-4)(n-5)}{2!}a_1^{n-6}a_2^2a_3^4$$

$$+\frac{n(n-1)(n-2)(n-3)}{4!}\frac{(n-4)(n-5)(n-6)(n-7)}{3!}a_1^{n-7}a_2^3a_3^4$$

$$+\frac{n(n-1)(n-2)(n-3)}{4!}(n-4)\frac{(n-5)(n-6)(n-7)}{4!}a_1^{n-8}a_2^4a_3^4$$

$$+\frac{n(n-1)(n-2)(n-3)}{4!}a_1^{n-4}*4a_3^3a_4+\frac{n(n-1)(n-2)(n-3)}{4!}(n-4)a_1^{n-5}a_2*4a_3^3a_4$$

$$+\frac{n(n-1)(n-2)(n-3)}{4!}\frac{(n-4)(n-5)(n-6)}{3!}a_1^{n-7}a_2^3*4a_3^3a_4$$

$$\left.+\frac{n(n-1)(n-2)(n-3)}{4!}\frac{(n-4)(n-5)(n-6)}{4!}(n-7)a_1^{n-7}a_2^4*4a_3^3a_4+\cdots\right)$$

$$+\left(\frac{n(n-1)(n-2)(n-3)}{4!}a_1^{n-4}*6a_3^2a_4^3+\frac{n(n-1)(n-2)(n-3)(n-4)}{4!}a_1^{n-5}a_2*6a_3^2a_4^3\right.$$

$$+\frac{n(n-1)(n-2)(n-3)}{4!}\frac{(n-4)(n-5)}{2!}a_1^{n-6}a_2^2*6a_3^2a_4^3$$

$$+\frac{n(n-1)(n-2)(n-3)}{4!}\frac{(n-4)(n-5)(n-6)}{3!}a_1^{n-7}a_2^3*6a_3^2a_4^3$$

$$\left.+\frac{n(n-1)(n-2)(n-3)}{4!}\frac{(n-4)(n-5)(n-6)(n-7)}{4!}a_1^{n-8}a_2^4*6a_3^2a_4^3+\cdots\right)$$

$$+\frac{n(n-1)(n-2)(n-3)}{4!}a_1^{n-5}a_2*4a_3a_4^3+\frac{n(n-1)(n-2)(n-3)}{4!}\frac{(n-4)(n-5)}{2!}a_1^{n-6}a_2^2*4a_3a_4^3$$

$$+\frac{n(n-1)(n-2)(n-3)}{4!}\frac{(n-4)(n-5)(n-6)}{3!}a_1^{n-6}a_2^3*4a_3a_4^3$$

$$\left.+\frac{n(n-1)(n-2)(n-3)}{4!}\frac{(n-4)(n-5)(n-6)(n-7)}{4!}a_1^{n-7}a_2^4*4a_3a_4^3+\cdots\right)$$

$$+\left(\frac{n(n-1)(n-2)(n-3)}{4!}a_1^{n-4}a_4^4+\frac{n(n-1)(n-2)(n-3)}{4!}(n-4)a_1^{n-5}a_2*a_4^4\right.$$

$$+\frac{n(n-1)(n-2)(n-3)}{4!}\frac{(n-4)(n-5)}{2!}a_1^{n-6}a_2^2a_4^4$$

$$+n(n-1)(n-2)(n-3)*\frac{(n-4)(n-5)(n-6)(n-7)}{4!}a_1^{n-7}a_2^{4}a_4^{4}$$
$$=\sum_{r=0}^{n}\sum_{r_{d_1}}^{n_{d_1}}\sum_{r_{d_2}}^{n_{d_2}}\binom{n}{r}a_1^{n-r}a_2^{r}a_3^{r_{d_1}}\binom{n_d}{r_d}a_2^{n_{d_1}}a_3^{r_{d_1}}a_4^{r_{d_2}}$$

More so,

$$\left[(a_1+a_2+a_3)+(a_4+a_5)\right]=\left[(a_1+a_2+a_3)^n+n(a_1+a_2+a_3)^{n-1}*(a_4+a_5)\right.$$
$$+\frac{n(n-1)}{2!}(a_1+a_2+a_3)^{n-2}(a_4+a_5)^2+\frac{n(n-1)(n-2)}{3!}(a_1+a_2+a_3)^{n-3}(a_4+a_5)^3$$
$$\left.+\frac{n(n-1)(n-2)(n-3)}{4!}(a_1+a_2+a_3)^{n-4}(a_4+a_5)^4+\cdots\right]$$
$$=\left\{\left[a_1+(a_2+a_3)\right]^n+n\left[a_1+(a_2+a_3)\right]^{n-1}(a_4+a_5)\right.$$
$$+\frac{n(n-1)}{2!}\left[a_1+(a_2+a_3)\right]^{n-2}(a_4+a_5)^2+\frac{n(n-1)(n-2)}{3!}\left[a_1+(a_2+a_3)\right]^{n-3}(a_4+a_5)^3$$
$$\left.+\frac{n(n-1)(n-2)(n-3)}{4!}\left[a_1+(a_2+a_3)\right]^{n-4}(a_4+a_5)^4+\cdots\right\}$$
$$=\{\lfloor a_1+(a_2+a_3)\rfloor\ +na_4\lfloor a_1+(a_2+a_3)\rfloor\ \ +na_5\lfloor a_1+(a_2+a_3)\rfloor$$
$$+a_4^{2}\frac{n(n-1)}{2!}\left[a_1+(a_2+a_3)\right]^{n-2}+2a_4a_5\frac{n(n-1)}{2!}\left[a_1+(a_2+a_3)\right]^{n-2}+a_5^{2}\frac{n(n-1)}{2!}\left[a_1+(a_2+a_3)\right]^{n-2}$$
$$+a_4^{3}\frac{n(n-1)(n-2)}{3!}\left[a_1+(a_2+a_3)\right]^{n-3}+3a_4^{2}a_5\frac{n(n-1)(n-2)}{3!}\left[a_1+(a_2+a_3)\right]^{n-3}$$
$$+3a_4a_5^{2}\frac{n(n-1)(n-2)}{3!}\left[a_1+(a_2+a_3)\right]^{n-3}+a_5^{3}\frac{n(n-1)(n-2)}{3!}\left[a_1+(a_2+a_3)\right]^{n-3}$$

$$+a_4^{4}\frac{n(n-1)(n-2)(n-3)}{4!}\left[a_1+(a_2+a_3)\right]^{n-4}+4a_4^{3}a_5\frac{n(n-1)(n-2)(n-3)}{4!}\left[a_1+(a_2+a_3)\right]^{n-4}$$
$$+6a_4^{2}a_5^{2}\frac{n(n-1)(n-2)(n-3)}{4!}\left[a_1+(a_2+a_3)\right]^{n-4}+4a_4a_5^{3}\frac{n(n-1)(n-2)(n-3)}{4!}\left[a_1+(a_2+a_3)\right]^{n-4}$$
$$\left.+a_5^{4}\frac{n(n-1)(n-2)(n-3)}{4!}\left[a_1+(a_2+a_3)\right]^{n-4}+\cdots\right\}$$
$$=\sum_{r=0}^{n}\sum_{r_{d_1}}^{n_{d_1}}\sum_{r_{d_2}}^{n_{d_2}}\sum_{r_{d_3}}^{n_{d_3}}\left[\binom{n}{r}a_1^{n-r}a_2^{r}a_3^{r_{d_1}}a_4^{r_{d_2}}\binom{n_d}{r_d}a_2^{n_{d_1}}a_3^{n_{d_2}}a_3^{r_{d_1}}a_4^{r_{d_2}}a_5^{r_{d_3}}\right]$$

Results on a Better Approach to Multinomial Expansion

We know that $[a_1 + a_2 + \cdots + a_n] = \left[\sum a_i\right]^n$

$$= \left[\left(\sum_{2i,i=1}^{2i,i=\frac{m}{2}} a_i\right) + \left(\sum_{2i,i=1}^{2i,i=m} a_i\right)\right]^n \tag{1.2.1}$$

where (1.2.1) is the Binomial expression of two multinomial expressions one even and the other odd multinomial. As such (1.2.1) becomes

$$\left[\left(\sum_{2i,i=1}^{2i,i=\frac{m}{2}} a_i\right) + \left(\sum_{2i,i=1}^{2i,i=m} a_i\right)\right]^n = \sum_{r=0}^{n}\binom{n}{r}\left(\sum_{2i,i=1}^{2i,i=\frac{m}{2}} a_i\right)^{n-r}\left(\sum_{2i,i=1}^{2i,i=m} a_i\right)^r \tag{1.2.2}$$

where the even term

$$\left(\sum_{2i,i=1}^{2i,i=\frac{m}{2}} a_i\right)^{n-r} = \sum_{k}^{n-r}\binom{n-r}{k}\left(\sum_{2i,i=1}^{2i,i=\frac{m}{2}} a_i\right)^{n-r-k}\left(\sum_{2i,i=1}^{2i,i=\frac{m}{2}} a_i\right)^k \tag{1.2.3}$$

and the odd term

$$\left(\sum_{2i,i=1}^{2i,i=m} a_i\right)^r = \sum_{q}^{r}\binom{r}{q}\left(\sum_{2i,i=1}^{2i,i=m} a_i\right)^{r-q}\left(\sum_{2i,i=1}^{2i,i=m} a_i\right)^q \tag{1.2.4}$$

Substituting (1.2.3) and (1.2.4) in (1.2.2) and in turn for (1.2.1) we have

$$\left[\left(\sum_{2i,i=1}^{2i,i=\frac{m}{2}} a_i\right) + \left(\sum_{2i,i=1}^{2i,i=m} a_i\right)\right]^n$$

$$= \sum\binom{n}{r}\binom{n-r}{k}\binom{r}{q}\left(\sum_{2i,i=1}^{2i,i=\frac{m}{2}} a_i\right)^{n-r-k}\left(\sum_{2i,i=1}^{2i,i=\frac{m}{2}} a_i\right)^k\left(\sum_{2i,i=1}^{2i,i=m} a_i\right)^{r-q}\left(\sum_{2i,i=1}^{2i,i=m} a_i\right)^q \tag{1.2.5}$$

Subsequent expansion and simplification of (1.2.5) results in the compact form stated below.

Theorem 1.2.1: The multinomial expansion for positive integral indices states that if

$$\left[\sum_{i=1}^{n} a_i\right]^n = \left[\left(\sum_{2i,i=1}^{2i,i=\frac{m}{2}} a_i\right) + \left(\sum_{2i,i=1}^{2i,i=m} a_i\right)\right]^n$$

then the compact form of the multinomial is

$$\sum_{k_1, k_2, \quad k_n}\binom{n}{k_1, k_2, \quad k_n} a_1^{k_1}\, a_2^{k_2} \quad a_n^{k_n}$$

while the coefficients $\binom{n}{k_1,\dots k_n}$ are called the multinomial coefficients which are computed by $\binom{n}{k_1,k_2,\dots k_n}=\frac{n!}{k_1!k_2!\dots k_n!}$.

Results on Consequences of Multinomial Theorem for Positive Integral Indices

Corollary 2.1: the multinomial expansion for negative integral indices (i.e. $n-$negative) is

$$(a_1+a_2+\cdots+a_n)^{-n}=\left[\sum_{i=1}^{m}a_i\right]^{-n}$$

$$=\left[\left(\sum_{2i,i=1}^{2i,i=\frac{m}{2}}a_i\right)+\left(\sum_{2i,i=1}^{2i,i=m}a_i\right)\right]^{-n}\sum\frac{\binom{n}{r}\binom{n-r}{k}\binom{r}{q}}{\left(\sum_{2i,i=1}^{2i,i=\frac{m}{2}}a_i\right)^{n-r-k}\left(\sum_{2i,i=1}^{2i,i=\frac{m}{2}}a_i\right)^{r}\left(\sum_{2i,i=1}^{2i,i=m}a_i\right)^{r-q}\left(\sum_{2i,i=1}^{2i,i=m}a_i\right)^{q}}$$

with the compact representation as

$$\sum_{k_1,k_2,\dots,k_n}\binom{n}{k_1,k_2,\dots k_n}a_1^{-k_1}a_2^{-k_2}\dots a_n^{-k_n}$$

while the coefficients $\binom{n}{k_1,k_2,\dots k_n}$, the multinomial coefficients which usually are computed by $\binom{n}{k_1,k_2,\dots k_n}=\frac{n!}{k_1!k_2!\dots k_n!}$.

Theorem 2.2: the multinomial theorem for functional indices (i.e. $n=\frac{p}{q}$) states that

$$\left[\sum_{i=1}^{m}a_i\right]^{\left(p/q\right)}=\left[\left(\sum_{2i,i=1}^{2i,i=\frac{m}{2}}a_i\right)+\left(\sum_{2i,i=1}^{2i,i=m}a_i\right)\right]^{\left(p/q\right)}$$

$$=\sum_{k_1,k_2,\dots,k_n}\binom{\left(p/q\right)}{k_1,k_2,\dots k_n}a_1^{k_1}a_2^{k_2}\dots a_n^{k_n}$$

while the multinomial coefficients are computed by $\binom{\left(p/q\right)}{k_1,k_2,\dots k_n}=\frac{\left(p/q\right)!}{k_1!k_2!\dots k_n!}$.

Theorem 2.3:(The Multi Binomial Expansion)

Consider the already existing Binomial expansion

$$[a_1 + a_2]^n = \sum_{k=0}^{\infty} \binom{r}{k} a_1^{r-k} a_2^{k}$$

then taking finite/infinite expansion of finite/infinite Binomial expansions, we obtain

$$(a_1 + a_2)^n (a_3 + a_4)^n \ldots (a_{i-1} + a_i)^n (a_{i+1} + a_{i+2})^n + \ldots$$

equals

$$(a_1 + a_2)^{n} \cdots (a_{n-1_d} + a_{n_d})^{n_d} = \sum_{k_1=0}^{n_1} \cdots \sum_{k_d=0}^{n_d} \binom{n_1}{k_1} a_1^{k_1} a_2^{n_1-k_1} \cdots \binom{n_d}{k_d} a_{n-1_d}^{k_d} a_{n_d}^{n_d-k_d}$$

This on a more compact presentation become

$$\prod_{i=1}^{\infty} (a_i + a_{i+1})^{\alpha} = \sum_{\upsilon \le \alpha} \binom{\alpha}{\upsilon} a^{\upsilon} a^{\alpha-\upsilon}$$

Main Results on the Multinomial Series from the Binomial Approach

The series $\sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{i_j}$ of real numbers is defined as a series of double sequence $\left\{ \sum_{j=1}^{n} a_j , \sum_{j=1}^{n} s_j \right\}$ satisfying the following conditions

$$\sum_{j=1}^{n} s_n = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{i_j}$$

where

$$\sum_{j=1}^{n} a_j = \sum_{i=1}^{n} s_j - \sum_{j=1}^{n} s_{j-1}$$

The series of numbers $\sum_{j=1}^{n} a_j$ is called the series of general terms of the series $\sum_{j=1}^{n} \sum_{i=1}^{m} a_{i_j}$ and $\sum_{j=1}^{n} s_j$ is called of *nth*particular sum of the series.

The series $\sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{i_j}$ is convergent is and only if its series of sequence of partial sums $\sum_{j=1}^{n} \{s_j\}$ is convergent to some series of numbers $\sum_{j=1}^{n} a_j^*$ say. In this case, the series of numbers $\sum_{j=1}^{n} a_j^*$ is called the sum of the series of series and we write $\sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{i_j} = \sum_{j=1}^{n} a_j^*$. The number

$\sum_{j=1}^{n} a_j^{*} - \sum_{j=1}^{n} s_{n_j} = \sum_{j=1}^{n}\sum_{i=1}^{\infty} a_{i_j}$ is called the remainder of the series of series or the tail which is denoted by

$$\sum_{j=1}^{n} R_{n_j} = \sum_{j=1}^{n} a_j^{*} - \sum_{j=1}^{n} s_{n_j} .$$

On the other hand a series of series is called divergent if it is not convergent.

Theorem 3.1:

a. The positive multinomial series $\sum_{j=1}^{\infty}\sum_{i=1}^{n}[a_i]^{n}$ for $n > 0$ is divergent provided $\left(\sum_{i=1}^{n} a_i\right)$ is not rational in which case the ratio $r > 1$.

b. The negative multinomial series $\sum_{j=1}^{\infty}\left[\sum_{i=1}^{n} a_i\right]^{-n}$ is an alternating series which may diverge or converge as the case may be.

c. The rational multinomial (i.e. for fractional indices) series is a convergent series if $n \geq 0$.

__Proof:__ The infinite integral sum $\sum_{j=1}^{n}\int_{b}^{\infty} f_n(t)dt$ is convergent if $\sum_{j=1}^{n} I_j(t) = \sum_{j=1}^{n}\int_{b}^{N} f_j(t)dt$ tends to a finite sum of limits as $N \to \infty$otherwise the integral is said to diverge. By the integral test, if f_n is a sequence of non-negative decreasing integral function such that

$f_n(m) = a_{n_m}(t)$ for all $m \in [b, \infty)$ then the series $\sum_{j=1}^{n}\sum_{i=1}^{m} a_{n_m}$ and the integral sum $\sum_{j=1}^{n}\int_{b}^{\infty} a_j(t)dt$ converge or diverge together. By this (a) is evident for if $a_i = f_i$ for $n > 0$, $r > 0$ in (a) the integral test above shows that $\sum_{j=1}^{n}\int a_j(t)dt > 0$ does not tend to a finite sum and hence does not converge and so is divergent. Following same integral test above we discover that the multinomial series in (b) proves alternating and hence may or may not converge but (c) above converges for $p < q$ in $n = p/q$.

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